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Time evolution of quantum systems with time-dependent Hamiltonian and the invariant Hermitian operator

Y-Z Lai†, J-Q Liang‡§‖, H J W Müller-Kirsten‡ and Jian-Ge Zhou‡

† Department for Basic Courses, Taiyuan Heavy Machinery Institute, Taiyuan, Shanxi, 030 024, People's Republic of China

‡ Department of Physics, University of Kaiserslautern, 67653 Kaiserslautern, Germany

§ Institute of Theoretical Physics, Shanxi University, Taiyuan, Shanxi 030 006, People's Republic of China

‖ Institute of Physics, Academia Sinica, Beijing 100 080, People's Republic of China

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Abstract. We study the time evolution of a class of exactly solvable time-dependent quantum systems with a time-dependent Hamiltonian given by a linear combination of $SU(1,1)$ and $SU(2)$ generators with the help of the invariant Hermitian operator. The exact common solutions of the Schrödinger equations for both the $SU(1,1)$ and $SU(2)$ systems are obtained in terms of eigenstates of the invariant operator. The adiabatic and non-adiabatic Berry phases are calculated with the exact solutions. Moreover, we derive an explicit time-evolution operator which is used to investigate the time-dependent two-photon squeezing states and $SU(2)$ squeezing states. The squeezing properties of the time-dependent $SU(1,1)$ coherent states are also discussed.

1. Introduction

The time-evolution of dynamical systems with an explicitly time-dependent Hamiltonian has attracted considerable attention because of its various applications. Lewis and Riesenfeld (RL) started investigating the dynamics and quantization of time-dependent systems long ago with the method of Hermitian invariants [1, 2]. If the time function of the Hamiltonian depends on a set of parameters, a cyclic evolution of the Hamiltonian in the parameter space leads to an additional phase which has a geometric significance and is known as Berry's phase [3]. Two models explaining the Berry phase have been studied extensively [4–12]. One of these is the time-dependent generalized harmonic oscillator, the Hamiltonian of which is a time-dependent linear function of the $SU(1,1)$ generator and the other is the two-level system with Hamiltonian consisting of the $SU(2)$ generator. In quantum optics, systems with time-dependent Hamiltonians are also of importance. The single-mode degenerate parametric amplifier with classical pumps is a well known example [13], and it is shown that a more general time-dependent Hamiltonian preserves the $SU(1,1)$ coherent states [14]. The localization of atomic states in a driving field is another example of time dependent systems [15]. The new interest in this time-dependent subject was motivated by the study of dissipative processes, which possess a time-dependent effective Hamiltonian† [16].

† For a simple model of dissipative systems, i.e. an harmonic oscillator coupled to the environment, the effective Hamiltonian is found to be the Hamiltonian of harmonic oscillator with a mass increasing with time. See, for example, [16].

Because of the time-dependence the Hamiltonian is no longer a conserved quantity, and the closed formula for time evolution of quantum states may be obtained with the help of invariant Hermitian operators [1, 2]. However, the derivation of the exact time-evolution operator for arbitrary time-dependent systems has not been given in the framework of invariant operator theory. In the present paper we present a construction of the invariant Hermitian operator in a manner as for both the $SU(1,1)$ and $SU(2)$ systems. An advantage of the invariant operator is that it allows one to obtain the exact solution of the Schrödinger equation in terms of eigenstates of the invariant operator as well as the time-evolution operator. Adiabatic and non-adiabatic Berry phases given in the literature are recovered with the common exact solution. Moreover, we investigate the time-dependent two-photon squeezing states and $SU(2)$ squeezing states using the time-evolution operator. The squeezing properties of $SU(1,1)$ coherent states are also studied as an extension of [14].

2. Invariant Hermitian operator, time evolution of quantum states and the Berry phase

The Hamiltonian which we consider is

$$\hat{H} = \omega(t)\hat{K}_0 + G(t)[\hat{K}_+ e^{i\varphi(t)} + \hat{K}_- e^{-i\varphi(t)}] \tag{2.1}$$

where $\omega(t)$, $G(t)$ and $\varphi(t)$ are arbitrary real functions of time. $\hat{K}_0$ is a Hermitian operator, while $\hat{K}_+ = (\hat{K}_-)^\dagger$.

The commutation relations of the operators are

$$[\hat{K}_0, \hat{K}_\pm] = \pm\hat{K}_\pm \qquad [\hat{K}_+, \hat{K}_-] = D\hat{K}_0. \tag{2.2}$$

The Lie algebra of $SU(2)$ and $SU(1,1)$ consists of the generators $\hat{K}_0$ and $\hat{K}_\pm$ corresponding to $D = 2$ and -2 in the commutation relations (2.2), respectively. The time evolution of quantum states is governed by the Schrödinger equation

$$i\frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle \tag{2.3}$$

where natural units $\hbar = c = 1$ are used throughout.

We start from an invariant Hermitian operator $\hat{I}$ which satisfies the condition

$$i\frac{\partial}{\partial t}\hat{I}(t) + [\hat{I}(t), \hat{H}(t)] = 0 \tag{2.4}$$

and $\hat{I}^\dagger = \hat{I}$. The key point of our method is the construction of the invariant operator $\hat{I}$ from the Hermitian operator $\hat{K}_0$ with a unitary transformation

$$\hat{I}(t) = \hat{R}(t)\hat{K}_0\hat{R}^\dagger(t) \tag{2.5}$$

where

$$\hat{R}(t) = \exp\left[\frac{\gamma(t)}{2}(\hat{K}_+ e^{-i\beta(t)} - \hat{K}_- e^{i\beta(t)})\right]. \tag{2.6}$$

The time-dependent real parameters $\gamma(t)$ and $\beta(t)$ are related to $G(t)$, $\varphi(t)$ and $\omega(t)$ in the Hamiltonian by the following auxiliary equations

$$\dot{\gamma} = 2G\sin(\varphi + \beta) \tag{2.7}$$

and

$$\frac{1}{\lambda}(\dot{\beta} - \omega)\sin\frac{\lambda}{2}\gamma = G\cos\frac{\lambda}{2}\gamma\cos(\varphi + \beta). \tag{2.8}$$

Here $G = \sqrt{2D}$ and the equations have been derived from equation (2.4) by the substitution of (2.5) and (2.6), and with the help of the following relations derived in Appendix A:

$$\hat{R}^\dagger(t)\hat{K}_+\hat{R}(t) = \hat{K}_+\cos^2\frac{\lambda}{4}\gamma - \hat{K}_-\mathrm{e}^{2\mathrm{i}\beta}\sin^2\frac{\lambda}{4}\gamma - \frac{D}{\lambda}\hat{K}_0\mathrm{e}^{\mathrm{i}\beta}\sin\frac{\lambda}{2}\gamma \tag{2.9}$$

$$\hat{R}^\dagger(t)\hat{K}_-\hat{R}(t) = \hat{K}_-\cos^2\frac{\lambda}{4}\gamma - \hat{K}_+\mathrm{e}^{-2\mathrm{i}\beta}\sin^2\frac{\lambda}{4}\gamma - \frac{D}{\lambda}\hat{K}_0\mathrm{e}^{-\mathrm{i}\beta}\sin\frac{\lambda}{2}\gamma \tag{2.10}$$

$$\hat{R}^\dagger(t)\hat{K}_0\hat{R}(t) = \hat{K}_0\cos\frac{\lambda}{2}\gamma + \frac{1}{\lambda}(\hat{K}_+\mathrm{e}^{-\mathrm{i}\beta} + \hat{K}_-\mathrm{e}^{\mathrm{i}\beta})\sin\frac{\lambda}{2}\gamma \tag{2.11}$$

$$\begin{aligned}\hat{R}^\dagger(t)\left[\mathrm{i}\frac{\partial}{\partial t}\hat{R}(t)\right] &= -2\hat{K}_0\dot{\beta}\sin^2\frac{\lambda}{4}\gamma + \hat{K}_+\mathrm{e}^{-\mathrm{i}\beta}\left(\mathrm{i}\frac{\dot{\gamma}}{2} + \frac{\dot{\beta}}{\lambda}\sin\frac{\lambda}{2}\gamma\right)\\ &\quad+\hat{K}_-\mathrm{e}^{\mathrm{i}\beta}\left(-\mathrm{i}\frac{\dot{\gamma}}{2} + \frac{\dot{\beta}}{\lambda}\sin\frac{\lambda}{2}\gamma\right).\end{aligned} \tag{2.12}$$

Using equations (2.7)–(2.12) one may check (see appendix B) that $\hat{I}$ satisfies the condition equation (2.4) and indeed is an invariant operator.

Let $|n\rangle$ be the eigenstate of $\hat{K}_0$ with eigenvalue K_n i.e.

$$\hat{K}_0|n\rangle = K_n|n\rangle. \tag{2.13}$$

The eigenstates of $\hat{I}(t)$ are obviously given by

$$\hat{I}(t)|n,t\rangle = K_n|n,t\rangle \qquad |n,t\rangle = \hat{R}(t)|n\rangle. \tag{2.14}$$

According to LR theory [1, 2], the general solution of the Schrödinger equation (2.3) is written as

$$|\psi(t)\rangle = \sum_n C_n \mathrm{e}^{\mathrm{i}\alpha_n(t)}|n,t\rangle \tag{2.15}$$

where the RL phase [1, 2] is

$$\begin{aligned}\alpha_n(t) &= \int_0^t \mathrm{d}t'\langle n,t'|\mathrm{i}\frac{\partial}{\partial t'} - \hat{H}(t')|n,t'\rangle\\ &= \int_0^t \mathrm{d}t'\langle n|\left[\hat{R}^\dagger(t')\mathrm{i}\frac{\partial}{\partial t}\hat{R}(t') - \hat{R}^\dagger(t')\hat{H}(t')\hat{R}(t')\right]|n\rangle\end{aligned} \tag{2.16}$$

is often considered as the non-adiabatic Berry phase if the parameters vary periodically. Using equations (9)–(12) we obtain the exact phase of the eigenstate

$$\alpha_n(t) = k_n\int_0^t \mathrm{d}t'[\omega(t') - D\Omega(t')] \tag{2.17}$$

where $\Omega(t)$ is given by equation (B6) in the appendix. The main motivation of our procedure in solving the $SU(1,1)$ and $SU(2)$ time-dependent systems with the invariant operator method is to find a more general and systematic way of dealing with Berry's phase. The original definition of Berry phase must be recovered in the adiabatic limit.

We now consider the adiabatic approximation where the terms containing time derivatives $\dot{r}$, $\dot{\beta}$ in the auxiliary equations (2.7) and (2.8) are neglected. We then have

$$\beta = 2n\pi - \varphi \qquad \frac{1}{\lambda}\omega\sin\frac{\lambda}{2}\gamma + G\cos\frac{\lambda}{2}\gamma = 0 \tag{2.18}$$

where n is integer.

It is easy to verify with the help of equations (2.7)–(2.12) that

$$\hat{R}^\dagger(t)\hat{H}(t)\hat{R}(t) = (\omega + D\Omega_0)\hat{K}_0 \tag{2.19}$$

where

$$\Omega_0 = -\frac{4\omega}{\lambda^2}\sin^2\frac{\lambda}{4}\gamma - \frac{2}{\lambda}G\sin\frac{\lambda}{2}\gamma. \tag{2.20}$$

Therefore $\hat{R}(t)|n\rangle$ becomes an instantaneous eigenstate of the Hamiltonian $\hat{H}$ with time-dependent eigenvalue $[\omega(t) + D\Omega_0(t)]K_n$ in the adiabatic limit†. As defined by Berry, the second term in equation (2.16) is the usual dynamical phase while the first term is the Berry phase denoted by γ_n. With the help of equation (2.12) and the commutation relation (2.2) the Berry phase is obtained as

$$\gamma_n(T) = \frac{4D}{\lambda^2}K_n\oint\sin^2\left(\frac{\lambda}{4}\gamma\right)\,\mathrm{d}\varphi \tag{2.21}$$

where T denotes the period of parameter variation. So far the Berry phase (2.21) is still a general formula for both $SU(2)$ and $SU(1,1)$ systems depending on $D = \pm 2$.

We now consider the $SU(1,1)$ case first where $D = -2$ and $\lambda = \pm 2\mathrm{i}$. The $SU(1,1)$ Lie algebra has a realization in terms of boson creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ such that

$$\hat{K}_0 = \tfrac{1}{2}(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}) \qquad \hat{K}_+ = \tfrac{1}{2}(\hat{a}^\dagger)^2 \qquad \hat{K}_- = \tfrac{1}{2}\hat{a}^2. \tag{2.22}$$

The Hamiltonian (1) then describes the generalized time-dependent harmonic oscillator. If ω and G are constant it reduces to a well known model in nonlinear quantum optics namely the single-mode degenerate parametric amplifier with classical pumps [13]. Substitution of $D = -2$, $\lambda = \pm 2\mathrm{i}$ and $K_N = \frac{1}{2}(N + \frac{1}{2})$ into equation (2.21) yields

$$\gamma_n(T) = (n + \tfrac{1}{2})\oint\sinh^2\frac{\gamma}{2}\,\mathrm{d}\varphi. \tag{2.23}$$

Solving the adiabatic auxiliary equation (2.18), the Berry phase can be found as an explicit formula depending only on the time-dependent parameters in the Hamiltonian, i.e.

$$\sinh^2\frac{\gamma}{2} = \frac{1}{2}\frac{\omega - \sqrt{\omega^2 - 4G^2}}{\sqrt{\omega^2 - 4G^2}}. \tag{2.24}$$

Considering the two-dimensional parameter space with vectors in polar coordinates

$$\boldsymbol{R} = (G(t)\sin\varphi(t), G(t)\cos\varphi(t)) \tag{2.25}$$

the Berry phase is seen to be

$$\gamma_n(T) = \tfrac{1}{2}(n + \tfrac{1}{2})\frac{\omega - \sqrt{\omega^2 - 4G^2}}{\sqrt{\omega^2 - 4G^2}}\oint\mathrm{d}\varphi \tag{2.26}$$

where $\oint\mathrm{d}\varphi = 2\pi$ for one period. The phase is due to the closed but not exact 1-form $\mathrm{d}\varphi$ in the multiply-connected two-dimensional parameter space [7].

For $D = 2$ with $\lambda = \pm 2$, Hamiltonian (1) possesses the symmetry of the dynamical group $SU(2)$. A spinning particle in a time-varying magnetic field is a practical example

† Another solution of equations (2.7) and (2.8) in the adiabatic approximation is

$$\beta = (2n+1)\pi - \varphi \qquad \frac{1}{\lambda}\pi\sin\frac{\lambda}{2}\gamma - G\cos\frac{\lambda}{2}\gamma = 0$$

which, however, does not lead to the desired result, namely $\hat{R}(t)|n\rangle$ is an instantaneous eigenstate of Hamiltonian $\hat{H}$, which is a crucial aspect of the original definition of Berry's phase. This may reflect a fact that the instantaneous stationary Schrödinger equation $\hat{H}(t)|n(t)\rangle = E_n(t)|n(t)\rangle$ is not gauge covariant [7, 8] but $\mathrm{i}\partial\psi(t)/\partial t = \hat{H}(t)\psi(t)$.

for this case. Let $\hat{K}_0 = \hat{J}_3$ and $\hat{K}_\pm = \hat{J}_\pm$. The eigenstate of $\hat{J}_3$ is $\hat{J}_3|j,n\rangle = n|j,n\rangle$. The Berry phase can be derived from the combining formula (2.21),

$$\gamma_n(T) = n\left(1 - \frac{\omega - \sqrt{\omega^2 - 4G^2}}{\sqrt{\omega^2 - 4G^2}}\right)\oint \mathrm{d}\varphi. \tag{2.27}$$

The Berry phase is again due to the closed but not exact 1-form $\mathrm{d}\varphi$ which contributes 2π for one period.

3. Time-evolution operator

With the help of equation (2.17), the general solution of the Schrödinger equation (2.15) can be rewritten as

$$|\psi(t)\rangle = \hat{R}(t)\mathrm{e}^{-\mathrm{i}\epsilon(t)\hat{K}_0}\hat{R}^\dagger(0)|\psi(0)\rangle \tag{3.1}$$

where

$$\epsilon(t) = \int_0^t \mathrm{d}t'\,[\omega(t') - D\Omega(t')]. \tag{3.2}$$

The time-evolution operator is obviously

$$\hat{U}(t,0) = \hat{R}(t)\mathrm{e}^{-\mathrm{i}\epsilon(t)\hat{K}_0}\hat{R}^\dagger(0) \tag{3.3}$$

and enjoys all properties of a unitary evolution operator:

$$\hat{U}^\dagger(t,0) = \hat{U}(0,t) = \hat{R}(0)\mathrm{e}^{\mathrm{i}\epsilon(t)\hat{K}_0}\hat{R}^\dagger(t) \tag{3.4}$$

$$\hat{U}(t_2,t_1) = \hat{U}(t_2,0)\hat{U}^\dagger(t_1,0). \tag{3.5}$$

In the Heisenberg picture the time evolution of any operator $\hat{A}$ is obtained with

$$\hat{A}(t) = \hat{U}^\dagger(t,0)\hat{A}\hat{U}(t,0). \tag{3.6}$$

Replacing D by ± 2 in $\hat{R}(t)$, we have the time-evolution operators for $SU(2)$ and $SU(1,1)$ systems, respectively. Some applications of the time-evolution operators are given in the following sections.

Before considering applications of the time-evolution operator we add a remark concerning previous work on these exactly solvable time-dependent quantum systems. The time-dependent Schrödinger equation is solved exactly with the elegant coherent state (CS) method [17, 18] for Hamiltonians which are a linear function of generators of dynamical (Lie) groups if its initial state is an arbitrary coherent state. The time-evolution operator for Hamiltonian linear combinations of the generators of $SU(1,1)$, $SU(2)$ and the Heisenberg–Weyl group is also given with the Wei–Norman method [19]. One advantage of the CS method is that it enables solutions of problems quite different in origin. Our approach is in the framework of LR theory for the purpose of dealing with the Berry phase. The construction of the LR invariant operator in the same manner as for both $SU(1,1)$ and $SU(2)$ systems may be in the spirit of the CS method. Different from the above methods, however, our exact solution equation (2.15), which is constructed in terms of eigenstates of the invariant operator, leads to the generalized Berry phase. The eigenstates of the invariant operator reduce to the instantaneous eigenstates of the time-dependent Hamiltonian in the adiabatic approximation so as to recover the original definition of Berry phase.

4. Time-dependent two-photon squeezing states and the squeezing properties of time-dependent $SU(1,1)$ coherent states

The boson realization of the $SU(1,1)$ Hamiltonian is

$$\hat{H} = \tfrac{1}{2}(\hat{a}^\dagger\hat{a} + \tfrac{1}{2})\omega(t) + \tfrac{1}{2}G(t)[(\hat{a}^\dagger)^2 e^{i\varphi(t)} + \hat{a}^2 e^{-i\varphi(t)}]. \tag{4.1}$$

In terms of the evolution defined in equation (3.6) where $D = -2$ and $\lambda = \pm 2i$ for the $SU(1,1)$ case, the time evolution of creation and annihilation operators is found with equation (3.9), i.e.

$$\hat{a}^\dagger(t) = \hat{a}^\dagger f_1^*(t) + \hat{a} f_2^*(t) \tag{4.1}$$

$$\hat{a}(t) = \hat{a} f_1(t) + \hat{a}^\dagger f_2(t) \tag{4.2}$$

where

$$f_1(t) = \cosh\frac{\gamma_0}{2}\cosh\frac{\gamma}{2}e^{-i\epsilon/2} - \sinh\frac{\gamma_0}{2}\sinh\frac{\gamma}{2}e^{i(\beta_0-\beta)}e^{i\epsilon/2} \tag{4.3}$$

$$f_2(t) = \cosh\frac{\gamma_0}{2}\sinh\frac{\gamma}{2}e^{-i\beta}e^{i\epsilon/2} - \sinh\frac{\gamma_0}{2}\cosh\frac{\gamma}{2}e^{-i\beta_0}e^{-i\epsilon/2} \tag{4.4}$$

with $\gamma_0 \equiv \gamma(t)|_{t=0}$, $\beta_0 = \beta(t)|_{t=0}$. The time-dependent parameters β, γ and ϵ are defined as before. In the above derivation, the following identities are used,

$$\hat{R}^\dagger(t)\hat{a}\hat{R}(t) = \hat{a}\cosh\frac{\gamma}{2} + \hat{a}^\dagger e^{-i\beta}\sinh\frac{\gamma}{2} \tag{4.5}$$

$$\hat{R}^\dagger(t)\hat{a}^\dagger\hat{R}(t) = \hat{a}^\dagger\cosh\frac{\gamma}{2} + \hat{a}e^{i\beta}\sinh\frac{\gamma}{2} \tag{4.6}$$

and

$$e^{i\epsilon\hat{K}_0}\hat{a}e^{-i\epsilon\hat{K}_0} = \hat{a}e^{-i\epsilon/2} \tag{4.7}$$

and

$$e^{i\epsilon\hat{K}_0}\hat{a}^\dagger e^{-i\epsilon\hat{K}_0} = \hat{a}^\dagger e^{i\epsilon/2} \tag{4.8}$$

where $\hat{K}_0 = \frac{1}{2}(\hat{a}^\dagger\hat{a} + \frac{1}{2})$. If ω and G are constant (while $\varphi = \omega t + \varphi$) the Hamiltonian (4.1) reduces to that for a degenerate parameter oscillator in nonlinear quantum optics which has been studied in detail by Gerry [13]. To compare with the degenerate parametric oscillator we assume that $\varphi(t)$ has the form

$$\varphi(t) = -\int_0^t dt'\,\omega(t') + \varphi_0 \tag{4.9}$$

φ_0 denoting the initial phase. With this choice of φ the auxiliary equations (2.7) and (2.8) have the special solution

$$\gamma(t) = 2\int_0^t dt'\,G(t') \qquad \beta(t) = \tfrac{1}{2}\pi - \varphi_0 + \int_0^t dt'\,\omega(t'). \tag{4.10}$$

Substitution of (4.10) into equation (B5) and equation (3.2) yields $\Omega(t) = 0$ and $\epsilon(t) = \int_0^t dt'\omega(t')$. Therefore

$$f_1(t) = \cosh\frac{\gamma}{2}e^{-i\epsilon/2} \qquad f_2(t) = -i\sinh\frac{\gamma}{2}e^{i(\epsilon-\varphi_0)/2}. \tag{4.11}$$

Following Gerry [14] we define the time-dependent Hermitian operator such that

$$\hat{x}_1 = \tfrac{1}{2}[\hat{a}e^{i\epsilon/2} + \hat{a}^\dagger e^{-i\epsilon/2}] \tag{4.12}$$

$$\hat{x}_2 = \frac{1}{2i}[\hat{a}e^{i\epsilon/2} - \hat{a}^\dagger e^{-i\epsilon/2}]. \tag{4.13}$$

The commutation relation is seen to be

$$[\hat{x}_1, \hat{x}_2] = \tfrac{1}{2}\mathrm{i} \tag{4.14}$$

from which follows the uncertainty relation

$$V(\hat{x}_1)V(\hat{x}_2) \geqslant \tfrac{1}{16} \tag{4.15}$$

where

$$V(\hat{x}_i) = \langle \hat{x}_i^2 \rangle - \langle \hat{x}_i \rangle^2 \tag{4.16}$$

if the initial state is prepared in a coherent state $|\alpha\rangle$ such that $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. Under time evolution generated by the Hamiltonian (4.1), the variances calculated with equation (4.16) using equations (4.1)–(4.4) are

$$V(\hat{x}_{1,2}) = \tfrac{1}{2}[\cosh\gamma \pm \sinh\varphi_0 \sinh\gamma]. \tag{4.17}$$

The $\pm$ signs correspond to operators $\hat{x}_1$ and $\hat{x}_2$, respectively. For the initial phase $\varphi_0 = \frac{1}{2}\pi$,

$$V(\hat{x}_1) = \tfrac{1}{4}\mathrm{e}^{\gamma(t)} \qquad V(\hat{x}_2) = \tfrac{1}{4}\mathrm{e}^{-\gamma(t)}. \tag{4.18}$$

We thus obtain a squeezed coherent state. Either quadrature $\hat{x}_1$ or $\hat{x}_2$ is squeezed depending on the sign of the time-dependent parameter $\gamma(t)$. Now consider the initial state being the $SU(1,1)$ coherent state. The Casimir invariant for $SU(1,1)$ Lie algebra consisting of $\hat{K}_0$ and $\hat{K}_\pm$ is given by

$$\hat{C} = \hat{K}_0^2 - \tfrac{1}{2}(\hat{K}_+\hat{K}_- - \hat{K}_-\hat{K}_+) \tag{4.19}$$

which for a unitary irreducible representation has eigenvalue $K(K-1)$, where K is the Bargman index. Confined to the representation of positive discrete series, the states $|n, K\rangle$ diagonalize the compact generator $\hat{K}_0$ such that $\hat{K}_0|n, K\rangle = (n+K)|n, K\rangle$. The operators $\hat{K}_+$ and $\hat{K}_-$ act as raising and lowering operators, respectively. For the boson realization of equation (2.22), $|n, \frac{1}{4}\rangle$ and $|n, \frac{3}{4}\rangle$ represent the photon number states with $2n$ and $(2n+1)$ photons, respectively. The $SU(1,1)$ coherent states are defined by

$$|\eta, K\rangle = D(\alpha)|0, K\rangle \tag{4.20}$$

where

$$\begin{aligned} D(\alpha) &= \exp(\alpha\hat{K}_+ - \alpha^*\hat{K}_-) \\ &= \mathrm{e}^{\eta\hat{K}_+}\mathrm{e}^{\xi\hat{K}_0}\mathrm{e}^{-\eta K_-} \end{aligned} \tag{4.21}$$

with $\alpha = -\frac{1}{2}\tau\mathrm{e}^{-\mathrm{i}\theta}$, $\eta = -\tanh(\tau/2)\mathrm{e}^{-\mathrm{i}\theta}$ and $\xi = \ln(1-|\eta|^2)$. Then $K = \frac{1}{4}$ and $\frac{3}{4}$ correspond to the even and odd coherent states, respectively. Suppose the initial state is the $SU(1,1)$ coherent state (4.20). Under time evolution generated by the Hamiltonian (4.1) the variances can be calculated in the same way using our time-evolution operator, while $\langle \hat{x}_{12} \rangle = 0$ in this case. Then

$$\begin{aligned} V(\hat{x}_{1,2}) = K\bigg[&\cosh\tau(\cosh\gamma \mp \sinh\gamma\sin\varphi_0) \\ &\pm \frac{1}{2}\sinh\tau\left(\cosh^2\frac{\gamma}{2}\cos\theta - \sinh^2\frac{\gamma}{2}\cos(\theta - 2\varphi_0)\right) \\ &\pm \sinh\tau\sin\gamma\sin(\theta-\varphi_0)\bigg]. \end{aligned} \tag{4.22}$$

The desired squeezing properties may be obtained by adjusting the initial phase φ_0 and the parameters θ, τ. If $G(t)$, $\omega(t)$ in the Hamiltonian (4.1) are constant the result (4.22) reduces to that given in [14].

5. Time-dependent $SU(2)$ squeezing states

Replacing $\hat{K}_0$ and $\hat{K}_\pm$ by $\hat{J}_3$ and $\hat{J}_\pm$, respectively, and noting that $\lambda = \sqrt{2D} = 2$ in the $SU(2)$ case, the Hamiltonian we considered is

$$\hat{H}(t) = \omega(t)\hat{J}_3 + G(t)[\hat{J}_+ \mathrm{e}^{\mathrm{i}\varphi(t)} + \hat{J}_- \mathrm{e}^{-\mathrm{i}\varphi(t)}]. \tag{5.1}$$

Let $|j, m\rangle$ be the eigenstates of $\hat{J}_3$ and $\hat{J}^2 = \hat{J}_3^2 + \frac{1}{2}(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+)$. As a matter of practical interest we again consider the case $\varphi(t) = -\int_0^t \mathrm{d}t'\, \omega(t') + \varphi_0$ in the auxiliary equations (2.7) and (2.8) which provide the solution of $\gamma(t)$ and $\beta(t)$ given in equation (4.10). We then have $\Omega(t) = 0$ and $\epsilon(t) = \int_0^t \mathrm{d}t'\, \omega(t')$. Substitution of the above into equation (3.6) yields the time-evolution operator in the simple form

$$\hat{U}(t, 0) = \exp\left[\frac{\gamma}{2}(\hat{J}_+ \mathrm{e}^{-\mathrm{i}\beta} - \hat{J}_- \mathrm{e}^{\mathrm{i}\beta})\right] \mathrm{e}^{-\mathrm{i}\epsilon \hat{J}_3}. \tag{5.2}$$

Observing that

$$\mathrm{e}^{-\mathrm{i}\epsilon\hat{J}_3} \hat{J}_\pm \mathrm{e}^{\mathrm{i}\epsilon\hat{J}_3} = \hat{J}_\pm \mathrm{e}^{\pm\mathrm{i}\epsilon} \tag{5.3}$$

the time evolution of $\hat{J}_3$ and $\hat{J}_\pm$ is seen to be

$$\hat{J}_3(t) = \hat{J}_3 \cos\gamma + \tfrac{1}{2}\mathrm{i}\sin\gamma[\hat{J}_- \mathrm{e}^{-\mathrm{i}\varphi_0} - \hat{J}_+ \mathrm{e}^{\mathrm{i}\varphi_0}] \tag{5.4}$$

$$\hat{J}_\pm(t) = \mathrm{e}^{\pm\mathrm{i}\epsilon}\left[\hat{J}_\pm \cos^2\frac{\gamma}{2} - \hat{J}_\mp \sin^2\frac{\gamma}{2} \mathrm{e}^{\mp\mathrm{i}2\varphi_0} \mp \mathrm{i}\hat{J}_3 \sin\gamma\, \mathrm{e}^{\pm\mathrm{i}\varphi_0}\right]. \tag{5.5}$$

We again define the Hermitian operators $\hat{J}_1$ and $\hat{J}_2$

$$\hat{J}_1 = \tfrac{1}{2}[\hat{J}_+ \mathrm{e}^{-\mathrm{i}\epsilon} + \hat{J}_- \mathrm{e}^{\mathrm{i}\epsilon}] \tag{5.6}$$

$$\hat{J}_2 = \frac{1}{2\mathrm{i}}[\hat{J}_+ \mathrm{e}^{-\mathrm{i}\epsilon} - \hat{J}_- \mathrm{e}^{\mathrm{i}\epsilon}] \tag{5.7}$$

such that

$$[\hat{J}_1, \hat{J}_2] = \mathrm{i}\hat{J}_3 \tag{5.8}$$

from which follows the uncertainty relation

$$V(\hat{J}_1)V(\hat{J}_2) \geqslant \tfrac{1}{4}|\langle\hat{J}_3\rangle|^2 \tag{5.9}$$

where

$$V(J_i) = \langle J_i^2\rangle - \langle J_i\rangle^2. \tag{5.10}$$

Next we consider the system as being prepared in the eigenstate $|j, m\rangle$ initially. In terms of equations (5.6) and (5.7) the time evolution of the variances is obtained as

$$V_{j,m}(\hat{J}_1(t)) = \tfrac{1}{2}[j(j+1) - m^2][1 - \sin^2\varphi_0 \sin^2\gamma] \tag{5.11}$$

$$V_{j,m}(\hat{J}_2(t)) = \tfrac{1}{2}[j(j+1) - m^2][1 - \cos^2\varphi_0 \sin^2\gamma]. \tag{5.12}$$

The time evolution of the expectation value of $\hat{J}_3$ is

$$\langle\hat{J}_3(t)\rangle = m\cos\gamma. \tag{5.13}$$

For $m = \pm j$, $\varphi_0 = \frac{1}{2}\pi$,

$$V_{j,\pm j}(\hat{J}_1) = \tfrac{1}{2}j\cos^2\gamma \leqslant \tfrac{1}{2}j|\cos\gamma| \tag{5.14}$$

indicating a stable squeezing of $\hat{J}_1$ quadrature. The $\hat{J}_2$ quadrature is squeezed if $m = \pm j$ and $\varphi_0 = 0$. The minimum uncertainty $V(\hat{J}_1)V(\hat{J}_2) = 0$ is reached when $m = 0$ or $r(t) = (n + \frac{1}{2})\pi$. The basic concept of squeezing spin (or angular momentum) states has been given in [20]. Spin squeezing is an important problem. Since diverse physical systems can be described by the $SU(2)$ generators, apart from the real spin of particles and magnons, these systems are, for example, collective two-level atoms, cooper pairs in superconductors and macroscopic two-state systems, such as interferometers, and Josephson junctions.

When $\omega(t)$ is a constant and $j = \frac{1}{2}$ Hamiltonian (5.1) reduces to the equation for a single two-level atom driven by a classical field. The time-evolution operator derived in this paper may be useful for a detailed study of the time-evolution properties of the system, for instance the localization of atomic states due to the influence of the driving field [15].

6. Conclusion

Time evolution of $SU(1,1)$ and $SU(2)$ time-dependent quantum systems is solved in the framework using invariant Hermitian operators. The key point of our procedure is the construction of the invariant operator with equation (2.5), which is particularly useful in deriving the generalized Berry phase and for recovering the original definition of the Berry phase in the adiabatic limit. The time-evolution operator is also obtained with the invariant operator.

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Appendix A

A.1. Proof of equations (2.9)–(2.11)

We define the following functions of parameter s

$$T_1(s) = \hat{R}^\dagger(t)\hat{K}_0\hat{R}(t) \qquad T_2(s) = \hat{R}^\dagger(t)(\hat{K}_+\mathrm{e}^{-\mathrm{i}\beta} + \hat{K}_-\mathrm{e}^{\mathrm{i}\beta})\hat{R}(t) \tag{A1}$$

with $s = \gamma/2$ where γ is the parameter in $\hat{R}(t)$. Differentiation of T_1 and T_2 with respect to s leads to

$$\begin{aligned}\frac{\partial T_1}{\partial s} &= \hat{R}^\dagger(t)[\hat{K}_0, \hat{K}_+\mathrm{e}^{-\mathrm{i}\beta} - \hat{K}_-\mathrm{e}^{\mathrm{i}\beta}]\hat{R}(t)\\ &= \hat{R}^\dagger(t)[\hat{K}_+\mathrm{e}^{-\mathrm{i}\beta} + \hat{K}_-\mathrm{e}^{\mathrm{i}\beta}]\hat{R}(t) = T_2(s)\end{aligned} \tag{A2}$$

and

$$\frac{\partial T_2}{\partial s} = -2DT_1. \tag{A3}$$

Solving the ordinary differential equations (A2) and (A3) with respect to s we have

$$T_2(s) = c_1\mathrm{e}^{\mathrm{i}\lambda s} + c_2\mathrm{e}^{-\mathrm{i}\lambda s} \tag{A4}$$

$$T_2(s) = [-\mathrm{i}\lambda c_1\mathrm{e}^{\mathrm{i}\lambda s} + \mathrm{i}\lambda c_2\mathrm{e}^{-\mathrm{i}\lambda s}]/2D \tag{A5}$$

where $\lambda = \sqrt{2D}$.

Using the initial conditions $T_1(0) = \hat{K}_0$, $T_2(0) = \hat{K}_+ e^{-i\beta} + \hat{K}_- e^{i\beta}$ the integration constants are determined by

$$c_1 + c_2 = \hat{K}_+ e^{-i\beta} + \hat{K}_- e^{i\beta} \qquad c_1 - c_2 = \frac{2i}{\lambda} D\hat{K}_0. \tag{A6}$$

Noting that

$$\hat{R}^\dagger(t)[\hat{K}_+ e^{-i\beta} - \hat{K}_- e^{i\beta}]\hat{R}(t) = \hat{K}_+ e^{-i\beta} - \hat{K}_- e^{i\beta} \tag{A7}$$

equations (2.9)–(2.11) follow from (A4)–(A7).

A.2. Proof of equation (2.12)

$$\begin{aligned}\hat{T}^\dagger(t) i\frac{\partial}{\partial t}\hat{R}(t) &= \hat{R}^\dagger(t)\left[i\dot{\gamma}\frac{\partial}{\partial\gamma}\hat{R}(t) + \dot{\beta}\hat{R}^\dagger(t) i\frac{\partial}{\partial\beta}\hat{R}(t)\right] \\ &= i\dot{\gamma}[\hat{K}_+ e^{-i\beta} - \hat{K}_- e^{i\beta}] + \dot{\beta}\hat{R}^\dagger(t) i\frac{\partial}{\partial\beta}\hat{R}(t)\end{aligned} \tag{A8}$$

where

$$\dot{\gamma} = \frac{\partial\gamma}{\partial t} \qquad \dot{\beta} = \frac{\partial\beta}{\partial t}.$$

Again define a function of s

$$F(s) = \hat{R}^\dagger(t) i\frac{\partial}{\partial\beta}\hat{R}(t) \tag{A9}$$

with $s = \gamma/2$. Then

$$\frac{\partial F}{\partial s} = \hat{R}^\dagger(t)\left[i\frac{\partial}{\partial\beta}, \hat{K}_+ e^{-i\beta} - \hat{K}_- e^{i\beta}\right]\hat{R}(t). \tag{A10}$$

Since $[i\partial/\partial\beta, e^{-i\beta}] = e^{-i\beta}$ and $[i\partial/\partial\beta, e^{i\beta}] = -e^{i\beta}$, we have

$$\begin{aligned}\frac{\partial F}{\partial s} &= \hat{R}^\dagger(t)[\hat{K}_+ e^{-i\beta} + \hat{K}_- e^{i\beta}]\hat{R}(t) \\ &= T_2(s) = c_1 e^{i\lambda s} + c_2 e^{-i\lambda s}.\end{aligned} \tag{A11}$$

Integrating (A11) yields

$$F(s) = c_3 + \frac{c_1}{i\lambda} e^{i\lambda s} - \frac{c_2}{i\lambda} e^{-i\lambda s}. \tag{A12}$$

Since $F(0) = 0$, the new integration constant is

$$c_3 = (c_2 - c_1)/i\lambda \tag{A13}$$

where c_1 and c_2 are calculated from (A6). $F(s)$ is obtained by substitution of c_1, c_2 and c_3 into (A12). Then we obtain equation (2.12) by replacing $\hat{R}^\dagger i\partial\hat{R}(t)/\partial\beta$ in equation (A8) by $F(s)$.

Appendix B. Proof of $\hat{I}(t)$ being invariant

$$\begin{aligned} \mathrm{i}\frac{\partial}{\partial t}\hat{I}(t) &= \left[\mathrm{i}\frac{\partial}{\partial t}\hat{R}(t)\right]\hat{K}_0\hat{R}^\dagger(t) + \hat{R}(t)\hat{K}_0\left[\mathrm{i}\frac{\partial}{\partial t}\hat{R}^\dagger(t)\right] \\ &= \hat{R}\left[\hat{R}^\dagger\left(\mathrm{i}\frac{\partial}{\partial t}\hat{R}\right)\right]\hat{K}_0\hat{R}^\dagger + \hat{R}\hat{K}_0\hat{R}^\dagger\left[\hat{R}\left(\mathrm{i}\frac{\partial}{\partial t}\hat{R}^\dagger\right)\right]. \end{aligned} \tag{B1}$$

Using

$$\left(\mathrm{i}\frac{\partial}{\partial t}\hat{R}\right)\hat{R}^\dagger + \hat{R}\left(\mathrm{i}\frac{\partial}{\partial t}\hat{R}^\dagger\right) = 0$$

we have

$$\mathrm{i}\frac{\partial}{\partial t}\hat{I}(t) = \hat{R}\left[\hat{R}^\dagger\left(\mathrm{i}\frac{\partial}{\partial t}\hat{R}\right), \hat{K}_0\right]\hat{R}^\dagger. \tag{B2}$$

Since

$$\begin{aligned} [\hat{I}(t), \hat{H}(t)] &= [\hat{R}\hat{K}_0\hat{R}^\dagger, \hat{H}(t)] \\ &= \hat{R}[\hat{K}_0, \hat{R}^\dagger\hat{H}(t)\hat{R}]\hat{R}^\dagger. \end{aligned} \tag{B3}$$

Therefore

$$\mathrm{i}\frac{\partial}{\partial t}\hat{I}(t) + [\hat{I}(t), \hat{H}(t)] = \hat{R}\left[\hat{K}_0, \hat{R}^\dagger\hat{H}(t)\hat{R} - \hat{R}^\dagger\left(\mathrm{i}\frac{\partial}{\partial t}\hat{R}\right)\right]\hat{R}^\dagger. \tag{B4}$$

Substituting equations (2.9)–(2.12) into equation (B4) and using equation (2.7) and equation (2.8) we have

$$\mathrm{i}\frac{\partial}{\partial t}\hat{I}(t) + [\hat{I}(t), \hat{H}(t)] = \hat{R}(t)[\hat{K}_0, \{\omega(t) + D\Omega(t)\}\hat{K}_0]\hat{R}^\dagger(t) = 0. \tag{B5}$$

Thus $\hat{I}(t)$ indeed is an invariant operator, where

$$\Omega(t) = (\dot{\beta} - \omega)\frac{4}{\lambda^2}\sin^2\frac{\lambda}{4}\gamma - \frac{2}{\lambda}G\sin\frac{\lambda}{2}\gamma\cos(\varphi + \beta). \tag{B6}$$

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